ITEP/TH-14/96

hepth/xxxxmmm

Peierls model and vacuum structure in the N=2 supersymmetric field theories

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Abstract

We suggest the quasiparticle picture behind the integrable structure of N=2 SYM theory,which arises if the Lax operator is considered as a Hamiltonian for the fermionic system.We compare the meaning of BPS states with the one coming from the D-brane interpretation and give some evidence for the compositeness of the selfdual strings.The temperature phase transition with the disappearence of the mass gap is conjectured.

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1 Introduction

The impressive progress has been achieved in the description of the vacuum structure of the both field and string theories since the pioneering works of Seiberg and Witten [1].It turned out that the natural symmetry arguments fix the data necessary to describe the vacuum structure to be the peculiar Riemann surface and one form on it.These data provide the prepotential for the low-energy effective theory and the exact mass spectrum.However the dynamical mechanism responsible for the vacuum configuration is still obscure.

The same data allow the natural interpretation in terms of the integrable systems intimately related with these Riemann surfaces.The integrable system serving the pure N=2 SYM theory appeares to be periodical Toda chain [2],SYM with the adjoint matter- Calogero-Moser system [3] and SYM with the fundamental matter -integrable spin chains [4].Prepotentials appeared to be the logariphms of the quasiclassical τ-functions and one form is identified with the action variable for the corresponding integrable system.The reviews of the approach can be found in [5].Nevertheless in spite of the evident progress in the conceptual explanations of the objects which seemed to be misterious previously the dynamical derivation of the relevant integrable systems is absent.Therefore any complementary viewpoint on the problem at hands can shed new light on the vacuum dynamics.

In this letter we attempt to reformulate the problem in terms of the Peierls model known in the solid state physics .It describes the dynamics of the 1d fermions on the fluctuating lattice.Fortunately some exact results are known for this model both for the continuum [6] and discrete cases [7] and the Riemann surface (vacuum state in SYM theory) plays the role of the dispersion law for the quasiparticle.We consider the specific interaction in the Peierls model allowing the explicit comparison with the Toda chain. We also will give interpretation of 4d BPS states in the quasiparticle terms.With this interpretation the stringy picture of the BPS states [8] gets interpretation as a "string of the quasiparticles" filling the forbidden or allowed zones.The phase transition known in the Peierls model implies the existence of the same phenomena in 4d case between the phases with $\Lambda_{QCD} = 0$ and $\Lambda_{QCD} \neq 0$.

2 Peierls model

In this section we review the main facts about the Peierls model relevant for the further consideration.Initially it was formulated to describe the selfconsistent behaviour of 1d fermions interacting with the fluctuating lattice and was applied to the analysis of 1d superconductivity.The Coulomn interaction between the fermions is neglected,the fermions are assumed to be in the external field defined by the lattice state while the lattice dynamics itself is modified by the fermions.In what follows we will discuss both the continuum and discrete Peierls models.

Hamiltonian density for the simplest continuum model looks as follows

$$H_{con} = \Psi^{+}\sigma_3\partial_x\Psi + \Psi^{+}(\sigma_-\Delta^* - \sigma_+\Delta)\Psi + \Delta^2 \tag{1}$$

where Δ represents the lattice potential and in general situation we have the sum $\sum_n \kappa_n I_n$ with KdV potentials I_n instead of the single quadratic term.The components of the fermionic wave function $\Psi = (u, v)$ obey the

equations

$$
\begin{aligned}
\partial u_E-(\Delta^2+\partial\Delta)u_E&=E^2u_E\\
\partial v_E-(\Delta^2+\partial\Delta)v_E&=E^2v_E
\end{aligned}
\tag{2}
$$

For the discrete version one has [7]

$$
\begin{gathered}
H_{dis}=\sum_n \Psi_n^+ v_n\Psi_n+\Psi_n^+ c_n\Psi_{n+1}+\Psi_n^+ c_{n-1}\Psi_{n-1}+\sum_i \kappa_i I_i\\
c_n=exp(x_{n+1}-x_n)\\
I_0N=\sum_n lnc_n; I_2N=\sum_n (c_n^2+v_n^2)
\end{gathered}
\tag{3}
$$

where x_n are the lattice variables.

The first question to be addressed to the model is about its ground state.To obtain the ground state we have to minimize the Hamiltonian with respect to the fermionic and lattice variables.The variation over the fermionic variables results in the Lax equation for the Toda chain system

$$
c_n\Psi_{n+i}+c_{n-1}\Psi_{n-1}+v_n\Psi_n=E\Psi_n \tag{4}
$$

Variation over the lattice degrees of freedom gives rise to the system of the finite number of algebraic equations.These equations provide the explicit form of the Riemann surface which corresponds to the solution.In the simplest case of two first Toda Hamiltonians one has the following system [7]

$$
\begin{gathered}
\kappa_2=\frac{i}{2\pi}\oint_{e_1}^{\mu}\frac{dE}{\sqrt{R(E)}}\\
\oint_{e_1}^{\mu}\frac{(2E_1-s_1)dE}{\sqrt{R(E)}}=0\\
\kappa_0=\frac{i}{2\pi}\oint_{e_1}^{\mu}(E^2-\frac{s_1E}{2}+\frac{s_2}{2}-\frac{s_2^2}{8})\frac{dE}{\sqrt{R(E)}}\\
\rho=\frac{i}{2\pi}\oint_{e_1}^{\mu}(E+b)\frac{dE}{\sqrt{R(E)}}\\
R=\prod_{i=1}^{4}(E-e_i); s_1=\sum e_i; s_2=\sum_{i<j}e_ie_j
\end{gathered}
\tag{5}
$$

where b is defined from the proper normalization conditions and μ is the chemical potential.Let us note that the system above is the analogue of the Virasoro constraints in the matrix models.

It is also possible to write down the explicit solutions for the fermionic wave functions Ψ_n and the lattice variables x_n in terms of the elliptic functions

$$
\Psi_n(z)=b_n\frac{\sigma^n(z+z_0)\sigma(z-\alpha_n)}{\sigma^n(z-z_0)\sigma(z-\alpha)} \tag{6}
$$

where z_0 is the position of the pole of the elliptic function E(z) defined by

$$
\begin{gathered}
z=\oint_{e_1}^{E}\frac{dE}{\sqrt{R(E)}}\\
\alpha_n=2nz_0+\alpha
\end{gathered}
\tag{7}
$$

$$
x_n-na=ln\frac{\theta_4(\rho(n-\nu-\frac{1}{2})}{\theta_4(\rho(n-\nu+\frac{1}{2})} \tag{8}
$$

and ν reflects the degeneration of the ground state,ρ is the fermionic density.The relation between the modulus of the curve and the parameters of the model looks as follows

$$\mid \tau \mid \frac{\theta_2\theta_3\theta_4}{\theta_1(\frac{\rho}{2})} = exp(-a) \qquad (9)$$

where a is the average distance between the lattice cites.

The key feature of the solution is the appearence of the fermionic mass gap which substitutes the Λ_{QCD}.It can be also proved that the chemical potential for the fermions lies in the forbidden zone for the quasiparticles.To define the dispersion law for the quasiparticles we can consider the periodicity property of the fermionic wave function on the lattice

$$\Psi_{n+N}(E) = e^{iNp(E)}\Psi_n(E). \qquad (10)$$

The dependence of the quasimomentum p on the energy E provides the dispersion law which can be represented as a two dimensional Riemann surface Σ in 4d space (p,E) with complex p and E.Fermionic wave function is uniquely defined on this surface and the number of its zeros coincides with the genus of the curve.

Another important characteristic is the spectrum of the exitations.It can be divided in two classes.The first one corresponds to the exitations of the lattice and can be called phonon.Another gapless exitation corresponds to the charge density wave. The second type is fermionic one and strongly depends on the fermionic density.At large density one has the polyaron type state when fermions are localized on the configurations (in the continuum case)

$$u(x) = const - \frac{2\chi}{ch^2(\sqrt{\chi}x)} \qquad (11)$$

In the opposite limit of the small ρ one has the delocalized fermionic state and the lattice potential

$$u(x) = const + \chi cos(2\sqrt{\chi}x + \phi) \qquad (12)$$

where χ is some constant. It was also shown [6] that another important states can be discovered if the external magnetic field is applied.It turns out that these states with the magnetic quantum numbers have the localized energy levels in the forbidden zone.

Exact solution to the ground state implies the possibility to determine the temperature dependence of the mass gap.Indeed it was shown [6] that the mass gap for the fermions gets renormalized and disappears at some critical value T_c.Being translated to the form of the dispersion law it tells us that the Riemann surface degenerates to a sphere above the phase transition point.

3 Integrable structure of SUSY YM theories and the dispersion laws

In this section we compare the data governing the vacuum structure of SUSY YM theory and the one from the Peierls model.Low energy effective action in YM theory is fixed by the Riemann surface and holomorphic

differential defined on it.Prepotential $\mathcal{F}$ can be derived from the relations

$$a_{D_i} = \frac{d\mathcal{F}}{da_i}$$
$$a_i = \oint_{A_i} \lambda \tag{13}$$
$$a_{D_i} = \oint_{B_i} \lambda; \lambda = dS = Edp$$

where A_i and B_i are the cycles on the curve Σ.Generically the equation for the Toda curve of the length N reads [2]

$$y + y^{-1} = P_N(E) \tag{14}$$

However it was proved [7] that only one gap vacuum configuration is stable,therefore the generic polynomial denenerates. In what follows we will treat both Riemann surface and dS in terms of the quasiparticles.Namely Σ coincides with the dispersion law after the identification $y = exp(iNp(E))$ and dS with the differential of the quasiparticle energy [1]

Let us start with rather general arguments concerning the relation (15). We adopt the following viewpoint;Toda chain reflecting Seiberg-Witten solution is substituted by the system of fermions on the dynamical lattice and the Lax operator for the Toda system is assumed to coincide with the fermionic Hamiltonian.Therefore we would like to investigate the coupled system of fermions on the lattice.The meaning of the variables a_i as lattice action variables was established before [2] and now we are going to clarify the general meaning of (15) where according to [2] $\mathcal{F}$ is nothing but the logariphm of the quasiclassical ("averaged") τ function.

If we compare (15) with the wellknown relation

$$\Delta\theta_i = \frac{\partial < S(x_i, I_i) >}{\partial I_i} \tag{15}$$

where $< S >$ is the averaged action and $\Delta\theta_i$ -so called Hanney angle, the immediate identification of the latter with a_{D_i} arises.Remind that the Hanney angle is the quasiclassical analogue of the quantum Berry phase and relation between them reads as follows

$$\Delta\theta_i = \frac{\partial\gamma}{\partial n_i} + O(h) \tag{16}$$

where n_i appears in the Bohr-Sommerfeld quantization condition.It is worth noting that usually the nontrivial Hanney angles comes from the intersection of the isoenergetic surfaces where the auxiliary magnetic charges are distributed and this is in the qualitative agreement with the "magnetic" interpretation of a_{D_i} in 4d case.Note that nontriviality of the Hanney angle actually is dictated by the nontrivial "bundle of the action-angle variables" over the parameter space.The latter coincides now with the spectral curve.Note also that this remark is in the perfect agreement with the definition of the Hanney angle as the integral over the closed loop in the parameter space

$$\Delta\theta_i = \int dB < pdq > \tag{17}$$

where B (coordinate on the spectral curve in our case) is the external parameter.One more argument supporting our point of view is the convenient appearence of the Berry phase which is known to have the meaning of the

[1] note that recently the dispersion laws of the quasiparticles were used for the general classification of the low energy effective actions. [13]

effective action arising after the averaging over the fast degrees of freedom.Therefore the discussion above suggests that low energy effective action describes the lattice degrees of freedom after the averaging over the fermions indeed.

Let us now relate the parameters of the Peierls model with the ones of 4d theory.At first it is useful to find the length of the lattice.For this purpose remind that Toda chain can be derived from the Calogero-Moser system via the dimensional transmutation proceedure. This means that we add the one adjoint multiplet with the mass M to the N=2 YM Lagrangian.Integrable system behind this theory is known to be the Calogero-Moser one where M plays the role of the coupling constant g. We can consider the following limiting proceedure [3]

$$g = g_0 e^a$$
$$g \to \infty; g_0 fixed \tag{18}$$

where Toda chain variables ϕ_n are defined in terms of the Calogero particles x_n as

$$x_n = na + \phi_n \tag{19}$$

and a is the average distance between the cites.Therefore the total length can be identified with the coupling constant $\tau_0 = \frac{4\pi i}{\alpha^2} + \frac{\theta}{2\pi}$ taken at the UV scale.To get the interpretation of the another parameter of the Peierls model,namely the density of the fermions let us compare the solution above (9) and the SW one. Immediately one gets (we restrict ourselves by SU(2) case) for the fermionic density $\rho = \frac{|\tau(u)|}{4}$ therefore the number of the fermions in the continuum case equals to the ratio of the renormalized and UV coupling constants.For the discrete case the fermionic density can be read off from (9).

4 BPS states

One of the most intriguing questions which has to be answered in this approach is about the proper meaning of BPS states.We will show that it can be interpreted as the string of quasiparticles described above on the complexified Fermi surface.Let us remind that BPS states in N=2 YM theory have masses (for SU(2) case)

$$M_{n,m} = \mid na + ma_D \mid \tag{20}$$

so in the context of the Peierls model this formulae asquires the meaning of the energy of the fermions filling the allowed or forbidden zones.The "strings of the quasiparticles" are wrapped around the noncontractable cycles on the spectral surface and numbers n,m correspond to the number of the strings.It is useful to compare our string picture of BPS states with the one suggested in [8] where these strings were interpreted as the intersection of 2-branes which lie on 5-branes with the Riemann surface Σ.It was shown that these strings are selfdual in the sense of [10].

Given a quasiparticle picture we can discuss the composite nature of the selfdual strings.Indeed there are some conjectures about the composite structure of string in the literature [8, 12] and the elementary object was assumed to be 0-brane living on p-brane.This conjecture gets some evidence in our approuch.Riemann surface which is interpreted as a spectral curve for the integrable system has another interpretation as a world-volume(or part of it) of D-branes. Gauge $U(1)$ fields on the single D-brane with RR charge [9] or SU(N) for N coinciding

D-branes [11] provide the natural topological field theory [12] and therefore the dynamical degrees of freedom in the Hitchin approach.Moreover the open strings ending on the branes give rise to the Wilson line sources.With this identification we can assume that our fermions are the 0-branes on the spectral surface of the integrable system.

Let us emphasize that W-bosons and monopoles in SW solution also have to be treated as the composite objects built from the spectral fermions.

Now we present the argument that the system can undergo the temperature phase transition which resembles the deconfainment phase transition in QCD. The main point we would like to mention is that the mass gap analogous to the Λ_{QCD} disappeares above some critical temperature [6]. In what follows we will consider the simplest SU(2) case but all features are captured already is this situation.

The question can be formulated as follows;at what temperature the equations determining the vacuum state allow solution without the forbidden zone?After the simple calculation one gets the equation

$$8\pi T_c^{\frac{1}{2}} = \int_{z_1}^{\infty} dz \frac{thz}{z\sqrt{z-2z_1}} \tag{21}$$

$$2zT_c = E - \mu; 2z_1 T_c = E_1 - \mu$$

where μ is chemical potential,E_1 can be identified with the vacuum expectation value of the scalar field in YM theory.If $\mu - E_1 \gg T_c$ the critical temperature appeares to be

$$T_c \propto | E_1 - \mu | exp(-4\pi\sqrt{| E_1 - \mu |}) \tag{22}$$

Thus we see that the theory manifests the phase transition behaviour and the value of the critical temperature is proportional to Λ_{QCD} .

5 Conclusions

We have shown that one can formulate the quasiparticle picture behind the integrable structure in N=2 YM theories.In such approach Lax operator has the meaning of the Hamiltonian for the fermions interacting with the lattice degrees of freedom.BPS states including W-bosons and monopoles get interpretation as the strings of the quasiparticles. Spectral surface of the integrable system plays the role of the dispersion law for the quasiparticles. Moreover we provide some evidence for the compositeness of the selfdual strings which have been previously identified with BPS states in the membrane picture. Let us emphasize that the Riemann surface defining the ground state of the model appears in the dynamical way.

Let us make a few conjectures about the possible nature of the fermions playing the key role in our consideration.To get some feeling let us consider the effective mass of the quasiparticle.The velocity can be found as $v = \frac{\partial E}{\partial p}$ and an easy calculation shows that it vanishes at the branching points of the spectral curve, therefore at these points quasiparticle asquires the infinite mass.The appearence of the points where the quasiparticles have the infinite masses indicates the possibility for the so called fermion condensate to appear [14] which is known in the solid state physics in the strong coupling regime.If we conjecture that this phenomena and the fermion condensate formation in the QFT have the related origin one can look for the fermionic degrees of

freedom condensing in the SUSY vacuum. This is known to happen with gluinos.One more suggestive picture comes from the usual QCD where fermions develop the band energy structure in the background of the instanton-antiinstanton ensemble which results in the usual fermion condensate.Moreover one can speculate that the wellknown potential in the Chern number coordinate becomes the dynamical variable.To include into the game the matter degrees of freedom one has to allow for the fermions the jumps between all lattice cites for the adjoint matter [3] or to assign the spin degrees of freedom to the lattice for the fundamental matter [4]. However we can,t present now more rigorous arguments and hope to return to this point elsewhere.

6 Acknowledgements

I am indebted to I.Krichever,H.Leutwyler,A.Marshakov and N.Prokof,ev for the useful discussions.I thank H.Leutwyler for the hospitality at the Institute for Theoretical Physics at Bern University where the part of the work was done.

The work was supported by INTAS 1010–CT93–0023.

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